

ON CATEGORIES OF (φ, Γ) -MODULES

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ABSTRACT. Let K be a complete discretely valued field of mixed characteristics $(0, p)$ with perfect residue field. One of the central objects of study in p -adic Hodge theory is the category of continuous representations of the absolute Galois group of K on finite-dimensional \mathbb{Q}_p -vector spaces. In recent years, it has become clear that this category can be studied more effectively by embedding it into the larger category of (φ, Γ) -modules; this larger category plays a role analogous to that played by the category of vector bundles on a compact Riemann surface in the Narasimhan-Seshadri theorem on unitary representations of the fundamental group of said surface. This category turns out to have a number of distinct natural descriptions, which on one hand suggests the naturality of the construction, but on the other hand forces one to use different descriptions for different applications. We provide several of these descriptions and indicate how to translate certain key constructions, which were originally given in the context of modules over power series rings, to the more modern context of perfectoid algebras and spaces.

Throughout, let p be a prime number and let K be a p -adic field, by which we mean a complete discretely valued field of mixed characteristics $(0, p)$ and perfect residue field. For instance, K may be a finite extension of \mathbb{Q}_p , but we do not restrict to this case unless otherwise specified.

In p -adic Hodge theory, one studies the relationship between different cohomology theories associated to algebraic (and more recently analytic) varieties over K . For example, by analogy with the comparison between the Betti and de Rham cohomologies associated to a complex algebraic variety, one has a p -adic comparison isomorphism comparing the p -adic étale and algebraic de Rham cohomologies of a variety over K . (This result has a long, rich, and continuing history which we do not wish to review here; see the introductions of [47] and [9] for up-to-date synopses.)

Continuing with this analogy, just as one encapsulates the Betti–de Rham comparison isomorphism in the construction of a *Hodge structure* associated to a complex algebraic variety, one would like to encode the p -adic comparison isomorphism into an object associated to a K -variety that “remembers” certain cohomology groups and their associated structures. One reason to do this is to study situations where one has putative instances of the comparison isomorphism corresponding to varieties which are expected to exist but not yet constructed; this situation occurs naturally in the study of Shimura varieties [42].

A first approximation to such a package is Fontaine’s definition of a (φ, Γ) -module [23]. To a continuous representation of G_K on a finite-dimensional \mathbb{Q}_p -vector space, Fontaine associates a module over a certain commutative ring (described explicitly in terms of formal Laurent series; see §1) equipped with semilinear actions of certain endomorphisms of the base ring (the eponymous φ and Γ). The fact that the representation can be recovered from this

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module rests on two pivotal observations: first, one can describe representations of Galois groups of fields of characteristic p on finitely generated \mathbb{Z}_p -modules in a manner reminiscent of Artin–Schreier theory, as shown by Katz [27, 4.1.1]; second, the infinite cyclotomic extension $K(\mu_{p^\infty})$ has the same Galois group as the field of Laurent series over the residue field of $K(\mu_{p^\infty})$, as shown by Fontaine–Wintenberger [22].

Given a smooth proper K -variety, its p -adic étale cohomology admits a continuous G_K -action and thus may be fed into Fontaine’s (φ, Γ) -module functor. Fontaine had already established how to pass functorially from p -adic étale cohomology to de Rham cohomology, so in principle the same information is captured in the (φ, Γ) -module; however, there is no obvious way to convert the (φ, Γ) -module into de Rham cohomology without first passing back to étale cohomology.

This defect was subsequently remedied by work of Cherbonnier–Colmez [14] and Berger [3], which shows that Fontaine’s construction may be modified to use a different base ring in which formal Laurent series are replaced by rigid analytic functions on suitable annuli (again see §1 for precise definitions). This has the crucial advantage of making “evaluation at a point” make enough sense so that Fontaine’s étale–de Rham construction factors through the (φ, Γ) -module. Among the various applications of this refinement, we single out one which will be relevant later in our story: one can use (φ, Γ) -modules in this sense to give explicit formulas in Iwasawa theory, such as for the Bloch–Kato exponential function and the Perrin–Riou reciprocity map [4].

Here we arrive at the point of departure for this paper: the Iwasawa theory we just alluded to is nowadays retronymically called *cyclotomic* Iwasawa theory, because similar considerations have subsequently been applied to towers of number fields whose Galois groups are various (not necessarily commutative) p -adic Lie groups. However, the construction of (φ, Γ) -modules described above gives a privileged role to the cyclotomic towers, so it is not immediately evident that they can be used to study Iwasawa theory beyond the cyclotomic case. While some initial work in this direction does exist [7, 8, 45], our aim here is not to address this issue directly, but to instead work at a more foundational level: we describe (φ, Γ) -modules, and the constructions used in Berger’s explicit formulas, in terms that do not distinguish the cyclotomic tower from other p -adic Lie towers.

Before saying more about this, it is important to emphasize the fact that the functor from p -adic representations of G_K to (φ, Γ) -modules in the sense of Berger is only a full embedding, not an equivalence of categories. Using work of the first author [28], one may characterize the essential image of the functor (the so-called *étale* (φ, Γ) -modules) as the *semistable* objects for a suitable degree function; this immediately suggests an analogy with the Narasimhan–Seshadri theorem [44] identifying unitary representations of the fundamental group of a compact Riemann surface with polystable bundles. The computation of (continuous) Galois cohomology of representations in terms of (φ, Γ) -modules, worked out in Fontaine’s setting by Herr [24, 25], adapts to Berger’s setting and extends to non-étale objects by work of R. Liu [41]. These developments have led to several new applications of p -adic Hodge theory, such as the construction of a p -adic local Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ with good interpolation properties [16] and the study of the geometry of eigenvarieties via triangulations of (φ, Γ) -modules [39, 2, 11, 12]; this paper may have some relevance to such applications also, but we refrain from speculating on this point here.

A first step towards realizing our goal of getting the cyclotomic tower out of the theory of (φ, Γ) -modules is to describe the category in a more agnostic fashion. We introduce two of these: one by Berger in the language of *B-pairs* and one by Fargues–Fontaine in terms of vector bundles on certain one-dimensional noetherian schemes (sometimes called *Fargues–Fontaine curves*). The latter description arises very naturally within the geometric reinterpretation of p -adic Hodge theory in the language of *perfectoid spaces*, as in the work of Scholze [46, 47, 48] and Kedlaya–Liu [36, 37].

However, these descriptions alone do not suffice to expunge the cyclotomic tower from the theory of (φ, Γ) -modules from the point of view of applications to Iwasawa theory. This is because Berger’s explicit formulas depend crucially on a certain construction involving reduced traces in the cyclotomic tower, which ultimately manifests as an operator on the power series used in the definition of the base ring of (φ, Γ) -modules (the ψ map; see Definition 7.1). To eliminate this construction, we make crucial use of the theory of *arithmetic deformations* of (φ, Γ) -modules, as developed with L. Xiao in [39]; this allows us to exchange the explicit use of the cyclotomic tower in the classical construction of (φ, Γ) -modules for an arithmetic deformation parametrizing cyclotomic twists. This makes it natural to consider other deformations corresponding to other p -adic Lie groups, including nonabelian ones. A natural next step would be to try to interpret results from any of [7, 8, 45] in this framework, but we stop short of this point; see Remark 7.4 and §8 for a brief discussion.

Before concluding this introduction, we set a few running notations. Our primary model for these and other notations is [39].

Hypothesis 0.1. Throughout this paper, as in this introduction, let K be a complete discretely valued field of mixed characteristics with perfect residue field k and absolute Galois group G_K ; we do not assume k is finite (i.e., that K is a finite extension of \mathbb{Q}_p) unless explicitly specified. Put $F = W(k)[1/p]$ for $W(k)$ the ring of Witt vectors over k , so that K/F is a finite totally ramified extension. Let A be an affinoid algebra over \mathbb{Q}_p in the sense of Tate, rather than the more expansive sense of Berkovich.

1. THE ORIGINAL CATEGORY OF (φ, Γ) -MODULES

We begin by describing the original construction of the category of (φ, Γ) -modules, into which the category of continuous representations of G_K on finite-dimensional \mathbb{Q}_p -vector spaces embeds; this is most explicit in the case $K = F$. In preparation for our later discussion, we escalate the level of generality to accommodate representations valued in affinoid algebras.

Definition 1.1. Let $\mathbf{Rep}_A(G_K)$ denote the category of continuous representations of G_K on finite projective A -modules. With more work, it would be possible to consider also representations on arbitrary finitely generated A -modules; we will not attempt this here.

Definition 1.2. Let $\mathcal{R}_{F,A}^\infty$ be the ring of rigid analytic functions on the disc $|\pi| < 1$ over $F \hat{\otimes}_{\mathbb{Q}_p} A$. This ring is complete for the topology of uniform convergence on quasicompact subspaces (Fréchet topology). The ring admits a continuous endomorphism φ defined by the formula

$$(1.2.1) \quad \varphi \left(\sum_n c_n \pi^n \right) = \sum_n \varphi_F(c_n) ((1 + \pi)^p - 1)^n,$$

where φ_F denotes the A -linear extension of Witt vector Frobenius map on F . The group $\Gamma = \mathbb{Z}_p^\times$ also admits a continuous action on $\mathcal{R}_{F,A}^\infty$ defined by the formula

$$(1.2.2) \quad \gamma \left(\sum_n c_n \pi^n \right) = \sum_n c_n ((1 + \pi)^\gamma - 1)^n \quad (\gamma \in \Gamma)$$

under the interpretation of $(1 + \pi)^\gamma$ as the binomial series

$$(1 + \pi)^\gamma = \sum_{n=0}^{\infty} \frac{\gamma(\gamma - 1) \cdots (\gamma - n + 1)}{n!} \pi^n.$$

Note that the actions of φ and Γ commute.

Definition 1.3. Let $\mathcal{R}_{F,A}$ be the direct limit of the rings of rigid analytic functions on the annuli $* < |\pi| < 1$ over $F \hat{\otimes}_{\mathbb{Q}_p} A$. This ring is complete for the direct limit topology (in the category of locally convex topological \mathbb{Q}_p -vector spaces) induced by the topologies of uniform convergence on quasicompact subspaces (the *LF topology*). We extend the actions of φ and Γ on $\mathcal{R}_{F,A}^\infty$ to continuous actions on $\mathcal{R}_{F,A}$ using the same formulas (1.2.1), (1.2.2). Note that $\mathcal{R}_{F,A}$ is connected whenever A is.

Definition 1.4. A (φ, Γ) -module over $\mathcal{R}_{F,A}$ is a finite projective $\mathcal{R}_{F,A}$ -module M equipped with commuting semilinear actions of φ and Γ such that the action of Γ is continuous for the LF topology. Here by a *semilinear action* of φ on a module M , we mean a semilinear map $\varphi_M : M \rightarrow M$ induced by an isomorphism $\varphi^* M \rightarrow M$; note that the isomorphism property does not ensure that φ_M acts bijectively on M , as this already fails for $M = \mathcal{R}_{F,A}$ with the standard φ -action.

Let $\Phi\Gamma_{F,A}$ denote the category of (φ, Γ) -modules over $\mathcal{R}_{F,A}$, viewed as an exact tensor category with rank function $\text{rank}_F : \Phi\Gamma_{F,A} \rightarrow \text{Cont}(\text{Spec } A, \mathbb{Z})$ computing the rank of the underlying $\mathcal{R}_{F,A}$ -module.

We will establish the following result in §4.

Theorem 1.5. *There exists a full embedding $\mathbf{Rep}_A(G_F) \rightarrow \Phi\Gamma_{F,A}$.*

In the interim, let us see how this result can be used to define a corresponding category with F replaced by K .

Definition 1.6. Let $\mathcal{R}_{K,A} \in \Phi\Gamma_{F,A}$ be the object of rank $[K : F]$ corresponding to $\text{Ind}_{G_K}^{G_F} \rho_{\text{triv}}$ via Theorem 1.7. The canonical isomorphisms $\rho_{\text{triv}} \otimes \rho_{\text{triv}} \cong \rho_{\text{triv}}^\vee \otimes \rho_{\text{triv}} \cong \rho_{\text{triv}}$ then correspond to an associative morphism $\mu_K : \mathcal{R}_{K,A} \otimes \mathcal{R}_{K,A} \rightarrow \mathcal{R}_{K,A}$; this gives $\mathcal{R}_{K,A}$ the structure of a finite flat $\mathcal{R}_{F,A}$ -algebra equipped with continuous actions of φ and Γ .

Let $\Phi\Gamma_{K,A}$ be the category of pairs (M, μ) for which $M \in \Phi\Gamma_{F,A}$ and $\mu : \mathcal{R}_{K,A} \otimes_{\mathcal{R}_{F,A}} M \rightarrow M$ is a morphism which is associative with respect to μ_K , i.e., the compositions

$$\begin{aligned} \mathcal{R}_{K,A} \otimes \mathcal{R}_{K,A} \otimes M &\xrightarrow{\mu_K \otimes 1} M_K \otimes M \xrightarrow{\mu} M, \\ \mathcal{R}_{K,A} \otimes \mathcal{R}_{K,A} \otimes M &\xrightarrow{1 \otimes \mu} \mathcal{R}_{K,A} \otimes M \xrightarrow{\mu} M \end{aligned}$$

coincide. In other words, these are finite projective $\mathcal{R}_{K,A}$ -modules equipped with commuting semilinear continuous actions of φ and Γ . We again view $\Phi\Gamma_{K,A}$ as an exact tensor category with rank function $\text{rank}_K = \text{rank}_F / [K : F]$ computing the rank of the underlying $\mathcal{R}_{K,A}$ -module.

Let K' be a finite extension of K . Define the induction functor $\text{Ind} : \Phi\Gamma_{K',A} \rightarrow \Phi\Gamma_{K,A}$ and the restriction functor $\text{Res} : \Phi\Gamma_{K,A} \rightarrow \Phi\Gamma_{K',A}$ by restriction of scalars and extension of scalars, respectively, along the natural map $\mathcal{R}_{K,A} \rightarrow \mathcal{R}_{K',A}$.

We may then formally promote Theorem 1.5 as follows.

Theorem 1.7. *There exists a full embedding $\mathbf{Rep}_A(G_K) \rightarrow \Phi\Gamma_{K,A}$ compatible with induction and restriction on both sides.*

Definition 1.8. The category $\Phi\Gamma_{K,A}$ admits duals, and hence internal Homs: the dual of $M \in \Phi\Gamma_{K,A}$ is the module-theoretic dual $M^\vee = \text{Hom}_{\mathcal{R}_{K,A}}(M, \mathcal{R}_{K,A})$ with the actions of φ, Γ constructed so that the canonical $\mathcal{R}_{K,A}$ -linear morphism $M^\vee \otimes_{\mathcal{R}_{K,A}} M \rightarrow \mathcal{R}_{K,A}$ is a morphism in $\Phi\Gamma_{K,A}$. (Note that the definition of the φ -action on M^\vee depends on the fact that the action of φ corresponds to an isomorphism $\varphi^*M \rightarrow M$, not just an arbitrary $\mathcal{R}_{K,A}$ -linear morphism.) For M corresponding to $V \in \mathbf{Rep}_A(G_K)$ via Theorem 1.7, M^\vee corresponds to the contragredient representation V^\vee .

Let $\mathcal{R}_{K,A}(1)$ denote the object of $\Phi\Gamma_{K,A}$ corresponding to the cyclotomic character χ in $\mathbf{Rep}_A(G_K)$ via Theorem 1.7. Concretely, $\mathcal{R}_{K,A}(1)$ can be written as the free module of rank 1 on a generator ε satisfying

$$\varphi(\varepsilon) = \varepsilon, \quad \gamma(\varepsilon) = \chi(\gamma)\varepsilon \quad (\gamma \in \Gamma).$$

For $M \in \Phi\Gamma_{K,A}$, define the *Cartier dual*

$$M^* = M^\vee(1) = M^\vee \otimes_{\mathcal{R}_{K,A}} \mathcal{R}_{K,A}(1) \in \Phi\Gamma_{K,A};$$

for M corresponding to $V \in \mathbf{Rep}_A(G_K)$ via Theorem 1.7, M^* corresponds to the Cartier dual of V (i.e., the contragredient of V twisted by the cyclotomic character).

Remark 1.9. The description of $\Phi\Gamma_{K,A}$ given above is consistent with [32] but not with most older references. The reason is that even if $\text{Maxspec } A$ is connected, in general $\text{Maxspec } \mathcal{R}_{K,A}$ is not connected; it is more typical to replace it with one of its connected components, and to replace Γ with the stabilizer of that component. See Remark 4.2 and [32, Remark 2.2.12] for further discussion.

Remark 1.10. The base ring in Fontaine's original theory of (φ, Γ) -modules was not the ring $\mathcal{R}_{K, \mathbb{Q}_p}$, but rather the completion of the subring of elements of $\mathcal{R}_{K, \mathbb{Q}_p}$ which are bounded (meaning equivalently that their coefficients or their values are bounded). This ring cannot naturally be interpreted in terms of functions on a rigid analytic space.

2. INTERLUDE ON PERFECTOID FIELDS

In preparation for giving alternate descriptions of the category $\Phi\Gamma_{K,A}$, we introduce the basic theory of *perfectoid fields*, which subsumes the earlier theory of *norm fields* on which the classical theory of (φ, Γ) -modules is built; we briefly discuss the relationship with the older theory in Remark 2.20, deferring to [32] for more historical discussion. In the process, we must do a bit of extra work in order to accommodate the coefficient ring A .

Definition 2.1. Let L be a field containing K which is complete with respect to a nonarchimedean absolute value, denoted $|\cdot|$. Let \mathfrak{o}_L denote the valuation subring of L (i.e., elements of norm at most 1). We say L is *perfectoid* if L is not discretely valued and the Frobenius map on $\mathfrak{o}_L/(p)$ is surjective.

Example 2.2. Suppose that $K = F$ and let L be the completion of $K(\mu_{p^\infty})$. Then

$$\begin{aligned}\mathfrak{o}_L &\cong (W(k)[\zeta_p, \zeta_{p^2}, \dots]/(1 + \zeta_p + \dots + \zeta_p^{p-1}, \zeta_p - \zeta_{p^2}^p, \zeta_{p^2} - \zeta_{p^3}^p, \dots)_{(p)}^\wedge \\ \mathfrak{o}_L/(p) &\cong k[T_1, T_2, \dots]/(1 + T_1 + \dots + T_1^{p-1}, T_1 - T_2^p, T_2 - T_3^p, \dots),\end{aligned}$$

so L is perfectoid; the same will hold for general K by Theorem 2.4 below. For some more general results that subsume this example, see Remark 2.20 and Lemma 2.21.

Hypothesis 2.3. For the remainder of §2, let L be a perfectoid field.

Theorem 2.4. *Define the multiplicative monoids*

$$\mathfrak{o}_{L^\flat} = \varprojlim_{x \mapsto x^p} \mathfrak{o}_L, \quad L^\flat = \varprojlim_{x \mapsto x^p} L.$$

- (a) *There is a unique way to promote \mathfrak{o}_{L^\flat} and L^\flat to rings in such a way that the map $\mathfrak{o}_{L^\flat} \rightarrow L^\flat$ and the composition $\mathfrak{o}_{L^\flat} \rightarrow \mathfrak{o}_L \rightarrow \mathfrak{o}_L/(p)$ become ring homomorphisms. (The map $\mathfrak{o}_{L^\flat} \rightarrow \mathfrak{o}_L$ is multiplicative but not additive.)*
- (b) *The ring L^\flat is a perfect field. In addition, the function $L^\flat \rightarrow L \xrightarrow{|\cdot|} \mathbb{R}$ is an absolute value with respect to which L^\flat is complete with valuation subring \mathfrak{o}_{L^\flat} .*
- (c) *The field L^\flat is also perfectoid.*
- (d) *Any finite extension of L , equipped with the unique extension of the absolute value, is again perfectoid.*
- (e) *The functor $L' \mapsto L'^\flat$ defines an equivalence of categories between finite extensions of L and L^\flat , and thereby a canonical isomorphism $G_L \cong G_{L^\flat}$.*

Proof. See [32, §1] and references therein. □

Definition 2.5. Define the field L^\flat and equip it with an absolute value as per Theorem 2.4(b). For $r > 0$, let $W^r(L^\flat)$ be the set of $x = \sum_{n=0}^\infty p^n [\bar{x}_n] \in W(L^\flat)$ such that $p^{-n} |\bar{x}_n|^r \rightarrow 0$ as $n \rightarrow \infty$. By [36, Proposition 5.1.2], this set is a subring of $W(L^\flat)$ on which the function $|\cdot|_r$ defined by

$$\left| \sum_{n=0}^\infty p^n [\bar{x}_n] \right|_r = \max_n \{p^{-n} |\bar{x}_n|^r\}$$

is a complete multiplicative norm; this norm extends multiplicatively to $W^r(L^\flat)[p^{-1}]$. For $0 < s \leq r$, let $\tilde{\mathcal{R}}_L^{[s,r]}$ be the completion of $W^r(L^\flat)[p^{-1}]$ with respect to $\max\{|\cdot|_s, |\cdot|_r\}$, and put $\tilde{\mathcal{R}}_{L,A}^{[s,r]} = \tilde{\mathcal{R}}_L^{[s,r]} \hat{\otimes}_{\mathbb{Q}_p} A$. Let $\tilde{\mathcal{R}}_{L,A}^r$ be the inverse limit of the $\tilde{\mathcal{R}}_{L,A}^{[s,r]}$ over all $s \in (0, r)$, equipped with the Fréchet topology. Let $\tilde{\mathcal{R}}_{L,A}$ be the direct limit of the $\tilde{\mathcal{R}}_{L,A}^r$ over all $r > 0$, equipped with the locally convex direct limit topology (LF topology).

The notation is meant to suggest a strong analogy between (for example) the ring $\mathcal{R}_{K,A}^{[s,r]}$ of power series convergent on a (relative) closed annulus and the somewhat more mysterious ring $\tilde{\mathcal{R}}_{L,A}^{[s,r]}$. In fact, one may (somewhat imprecisely) think of the latter as consisting of certain “Laurent series in p with Teichmüller coefficients”; this point of view is pursued in [31] to express certain geometric consequences. In the case where A is a field, a simultaneous development of the ring-theoretic properties of $\mathcal{R}_{K,A}^{[s,r]}$ and $\tilde{\mathcal{R}}_{L,A}^{[s,r]}$ can be found in [29].

For A a field, the following is a consequence of [37, Theorem 3.5.8].

Lemma 2.6. *Let L' be the completion of a (possibly infinite) Galois algebraic extension of L with Galois group G . Then for $0 < s \leq r$, the functor from finite projective $\tilde{\mathcal{R}}_{L,A}^{[s,r]}$ -modules to finite projective $\tilde{\mathcal{R}}_{L',A}^{[s,r]}$ -modules equipped with continuous semilinear G -actions is an equivalence of categories.*

Proof. We first check full faithfulness. Let M, N be two finite projective $\tilde{\mathcal{R}}_{L,A}^{[s,r]}$ -modules and put $P = M^\vee \otimes N$. Let M', N', P' be the respective base extensions of M, N, P to $\tilde{\mathcal{R}}_{L',A}^{[s,r]}$, equipped with the induced G -actions. We then have maps

$$P \cong \mathrm{Hom}(M, N) \rightarrow \mathrm{Hom}_G(M', N') \cong (P')^G,$$

so to check full faithfulness we need only check that $P \rightarrow (P')^G$ is an isomorphism. By writing P as a direct summand of a finite free module, this reduces immediately to checking that

$$(\tilde{\mathcal{R}}_{L',A}^{[s,r]})^G = \tilde{\mathcal{R}}_{L,A}^{[s,r]}.$$

For $A = \mathbb{Q}_p$, this equality is a consequence of [36, Theorem 9.2.15]; we may deduce the general case from this by constructing a Schauder basis for A over \mathbb{Q}_p , as in [10, Proposition 2.7.2/3] or [36, Lemma 2.2.9(b)].

We next check essential surjectivity. Let M' be a finite projective $\tilde{\mathcal{R}}_{L',A}^{[s,r]}$ -module equipped with a continuous semilinear G -action. The G -action may then be described in terms of an isomorphism ι between the two base extensions of M' to $\tilde{\mathcal{R}}_{L',A}^{[s,r]} \hat{\otimes}_{\tilde{\mathcal{R}}_{L',A}^{[s,r]}} \tilde{\mathcal{R}}_{L',A}^{[s,r]}$; note that ι obeys a cocycle condition expressing the compatibility of the G -action with composition in the group G . Specifying ι involves only finitely elements of the ring $\tilde{\mathcal{R}}_{L',A}^{[s,r]}$, so it may be realized over some subfield of L' which is the completion of an algebraic extension of L of at most countable degree; we may thus assume that L' itself has this form. From the proof of [36, Theorem 9.2.15], we see that $\tilde{\mathcal{R}}_{L',\mathbb{Q}_p}^{[s,r]}$ splits in the category of Banach modules over $\tilde{\mathcal{R}}_{L,\mathbb{Q}_p}^{[s,r]}$; by tensoring with A , we see that $\tilde{\mathcal{R}}_{L',A}^{[s,r]}$ splits in the category of Banach modules over $\tilde{\mathcal{R}}_{L,A}^{[s,r]}$. This means that $\tilde{\mathcal{R}}_{L',A}^{[s,r]} \rightarrow \tilde{\mathcal{R}}_{L,A}^{[s,r]}$ is a universally injective morphism in the category of Banach modules over $\tilde{\mathcal{R}}_{L,A}^{[s,r]}$, so we may apply a general descent theorem of Joyal–Tierney [26] (compare [37, Lemma 1.2.17]) to descend M' to a finite projective $\tilde{\mathcal{R}}_{L,A}^{[s,r]}$ -module M . (See also [50, Tag 08WE] for a more elementary treatment of the corresponding descent statement for ordinary modules over a ring, whose proof may be emulated for Banach modules.) \square

Remark 2.7. It is shown in [33, Theorem 3.2] that the rings $\tilde{\mathcal{R}}_{L,\mathbb{Q}_p}^{[s,r]}$ are *really strongly noetherian*, that is, any affinoid algebra over such a ring (even in the sense of Berkovich) is noetherian. In particular, the rings $\tilde{\mathcal{R}}_{L,A}^{[s,r]}$ are really strongly noetherian; consequently, they satisfy the analogues of Tate’s acyclicity theorem [34, Theorem 7.14, Theorem 8.3] and Kiehl’s theorem on the characterization of coherent sheaves [34, Theorem 8.16].

Definition 2.8. Denote by φ the following maps induced by the Witt vector Frobenius map on $W(L)$:

$$W^r(L) \rightarrow W^{r/p}(L), \quad \tilde{\mathcal{R}}_{L,A}^{[s,r]} \rightarrow \tilde{\mathcal{R}}_{L,A}^{[s/p,r/p]}, \quad \tilde{\mathcal{R}}_{L,A}^r \rightarrow \tilde{\mathcal{R}}_{L,A}^{r/p}, \quad \tilde{\mathcal{R}}_{L,A} \rightarrow \tilde{\mathcal{R}}_{L,A}.$$

A φ -module over $\tilde{\mathcal{R}}_{L,A}$ is a finite projective $\tilde{\mathcal{R}}_{L,A}$ -module equipped with a semilinear φ -action; unlike for (φ, Γ) -modules, this action is necessarily bijective (because the same is true of the maps φ displayed above). Let $\Phi\mathbf{Mod}_{L,A}$ be the category of φ -modules over $\tilde{\mathcal{R}}_{L,A}$.

Lemma 2.9. *For any r, s with $0 < s \leq r/p$, we have*

$$\ker(\varphi - 1 : \tilde{\mathcal{R}}_{L,A}^{[s,r]} \rightarrow \tilde{\mathcal{R}}_{L,A}^{[s,r/p]}) = A.$$

In particular, we have $\tilde{\mathcal{R}}_{L,A}^\varphi = A$.

Proof. Again using [36, Lemma 2.2.9(b)], we reduce to the case $A = \mathbb{Q}_p$, for which see [36, Corollary 5.2.4]. \square

Lemma 2.10. *For any r, s with $0 < s \leq r/p$, the following categories are canonically equivalent:*

- (a) *the category $\Phi\mathbf{Mod}_{L,A}$;*
- (b) *the category of finite projective $\tilde{\mathcal{R}}_{L,A}^r$ -modules M equipped with isomorphisms*

$$\varphi^* M \cong M \otimes_{\tilde{\mathcal{R}}_{L,A}^r} \tilde{\mathcal{R}}_{L,A}^{r/p};$$

- (c) *the category of finite projective $\tilde{\mathcal{R}}_{L,A}^{[s,r]}$ -modules M equipped with isomorphisms*

$$\varphi^* M \otimes_{\tilde{\mathcal{R}}_{L,A}^{[s/p, r/p]}} \tilde{\mathcal{R}}_{L,A}^{[s,r/p]} \cong M \otimes_{\tilde{\mathcal{R}}_{L,A}^{[s,r]}} \tilde{\mathcal{R}}_{L,A}^{[s,r/p]}.$$

Proof. The functor from (b) to (a) is base extension from $\tilde{\mathcal{R}}_{L,A}^r$ to $\tilde{\mathcal{R}}_{L,A}$. The fact that it is an equivalence is an easy consequence of the bijectivity of the action of φ on objects of $\Phi\mathbf{Mod}_{L,A}$.

The functor from (b) to (c) is base extension from $\tilde{\mathcal{R}}_{L,A}^r$ to $\tilde{\mathcal{R}}_{L,A}^{[s,r]}$. To prove that it is an equivalence, note first that (c) is formally equivalent to the same category with r, s replaced by $r/p, s/p$. Using Remark 2.7, we see additionally that (c) is equivalent to the same category no matter what values of r, s are used. We may then check the equivalence between (b) and (c) by imitating the proof of [39, Proposition 2.2.7]. \square

Definition 2.11. For $M \in \Phi\mathbf{Mod}_{L,A}$ and $n \in \mathbb{Z}$, define the twist $M(n)$ to have the same underlying module as M , but with the action of φ multiplied by p^{-n} .

Lemma 2.12. *For $M \in \Phi\mathbf{Mod}_{L,A}$, there exists $n_0 \in \mathbb{Z}$ such that for all $n \geq n_0$, $\varphi - 1$ is surjective on $M(n)$ and its kernel generates $M(n)$ as a $\tilde{\mathcal{R}}_{L,A}$ -module.*

Proof. The case $A = \mathbb{Q}_p$ is treated in [36, Proposition 6.2.2, Proposition 6.2.4]; the same proofs carry over to the general case. \square

Theorem 2.13. *Let \mathbb{C}_L be a completed algebraic closure of L . Then the formula*

$$V \mapsto (V \otimes_A \tilde{\mathcal{R}}_{\mathbb{C}_L, A})^{G_L}$$

defines a full embedding $\mathbf{Rep}_A(G_L) \rightarrow \Phi\mathbf{Mod}_{L,A}$. (We will discuss the essential image of this functor in §3.)

Proof. The target of this functor is in $\Phi\mathbf{Mod}_{L,A}$ thanks to Lemma 2.6 and Lemma 2.10. If $V \in \mathbf{Rep}_A(G_L)$ corresponds to $M \in \Phi\mathbf{Mod}_{L,A}$, then by Lemma 2.6 there is a canonical (φ, G_L) -equivariant isomorphism

$$V \otimes_A \tilde{\mathcal{R}}_{\mathbb{C}_L,A} \cong M \otimes_{\tilde{\mathcal{R}}_{L,A}} \tilde{\mathcal{R}}_{\mathbb{C}_L,A}.$$

By Lemma 2.9, we may take φ -invariants to obtain an isomorphism

$$V \cong (M \otimes_{\tilde{\mathcal{R}}_{L,A}} \tilde{\mathcal{R}}_{\mathbb{C}_L,A})^\varphi;$$

from this we see that the functor $V \mapsto M$ is fully faithful. \square

Definition 2.14. There is a canonical surjection $\theta : W(\mathfrak{o}_{L^\flat}) \rightarrow \mathfrak{o}_L$ whose kernel is a principal ideal; see [32, §1] for the construction. By [36, Lemma 5.5.5], for any interval $[s, r]$ containing 1, this map extends to a surjection $\theta : \tilde{\mathcal{R}}_{L,A}^{[s,r]} \rightarrow L \hat{\otimes}_{\mathbb{Q}_p} A$.

We now introduce a geometric construction developed in great detail by Fargues and Fontaine [20]; see [18, 19] for expository treatments.

Definition 2.15. Define the graded ring

$$P_{L,A} = \bigoplus_{n=0}^{\infty} P_{L,A,n}, \quad P_{L,A,n} = \tilde{\mathcal{R}}_{L,A}^{\varphi=p^n},$$

and put $X_{L,A} = \mathrm{Proj}(P_{L,A})$. Let $\mathbf{VB}_{L,A}$ be the category of quasicoherent locally finite free sheaves (or for short *vector bundles*) on $X_{L,A}$.

Example 2.16. For any x in the maximal ideal of \mathfrak{o}_{L^\flat} , the sum $\sum_{n \in \mathbb{Z}} p^{-n} [\bar{x}^{p^n}]$ converges to a nonzero element of $P_{L,\mathbb{Q}_p,1}$.

Definition 2.17. The map θ defines a closed immersion $\mathrm{Spec}(L \hat{\otimes}_{\mathbb{Q}_p} A) \rightarrow X_{L,A}$; let $Z_{L,A}$ be the resulting closed subscheme of $X_{L,A}$, and let $U_{L,A}$ be the complement of $Z_{L,A}$ in $X_{L,A}$. By [36, Lemma 8.9.3], $U_{L,A}$ is affine and $Z_{L,A}$ is contained in an open affine subspace of $X_{L,A}$; consequently, we may complete $X_{L,A}$ along $Z_{L,A}$ to get another affine scheme $\hat{Z}_{L,A}$. Let $\mathbf{B}_{e,L,A}$, $\mathbf{B}_{\mathrm{dR},L,A}^+$, $\mathbf{B}_{\mathrm{dR},L,A}$ be the respective coordinate rings of the affine schemes

$$U_{L,A}, \hat{Z}_{L,A}, U_{L,A} \times_{\mathrm{Proj}(P_{L,A})} \hat{Z}_{L,A}.$$

Let $\mathbf{BPair}_{L,A}$ be the category of glueing data for finite projective modules with respect to the diagram

$$\mathbf{B}_{e,L,A} \rightarrow \mathbf{B}_{\mathrm{dR},L,A} \leftarrow \mathbf{B}_{\mathrm{dR},L,A}^+.$$

Theorem 2.18. *The categories $\Phi\mathbf{Mod}_{L,A}$, $\mathbf{BPair}_{L,A}$, and $\mathbf{VB}_{L,A}$ are canonically equivalent.*

Proof. The categories $\mathbf{BPair}_{L,A}$ and $\mathbf{VB}_{L,A}$ are equivalent by the Beauville-Laszlo theorem [1] applied to the coordinate ring of some open affine subscheme of $X_{L,A}$ containing $Z_{L,A}$. We construct the functor from $\mathbf{VB}_{L,A}$ to $\Phi\mathbf{Mod}_{L,A}$ as in [36, Definition 6.3.10]. Choose $\mathcal{F} \in \mathbf{VB}_{L,A}$. For each $f \in P_{L,A}$ which is homogeneous of positive degree, we have an open affine subscheme of $X_{L,A}$ with coordinate ring $\tilde{\mathcal{R}}_{L,A}[f^{-1}]^\varphi$; we may thus take sections of \mathcal{F} to obtain a finite projective module over this ring. By base extension, we obtain a finite projective module over $\tilde{\mathcal{R}}_{L,A}[f^{-1}]$ equipped with a semilinear φ -action. By [36, Lemma 6.3.7],

the possible values of f generate the unit ideal in $\tilde{\mathcal{R}}_{L,A}$, so we may glue on $\mathrm{Spec}(\tilde{\mathcal{R}}_{L,A})$ to obtain an object of $\Phi\mathbf{Mod}_{L,A}$.

In the other direction, for $M \in \Phi\mathbf{Mod}_{L,A}$, we may view $\bigoplus_{n=0}^{\infty} M(n)\varphi$ as a graded module over $P_{L,A}$, and then form the associated quasicoherent sheaf, which we must show is a vector bundle. Using Lemma 2.12, this follows as in the proof of [36, Theorem 6.3.12]. \square

Remark 2.19. Throughout this remark, assume that A is a field. In this setting, the concept of B -pairs was introduced by Berger [5] in a purely algebraic fashion, without reference to the schemes defined in Definition 2.17. Therein, the ring $\mathbf{B}_{e,L,A}$ appears in connection with Fontaine’s crystalline period ring $\mathbf{B}_{\mathrm{crys}}$.

The scheme $X_{L,A}$ introduced by Fargues–Fontaine is in some sense a “complete curve”: in particular, it is a regular one-dimensional noetherian scheme. The space $P_{L,A,n}$ constitutes the sections of the n -th power of a certain ample line bundle on this scheme. This scheme admits something resembling an analytification in the category of adic spaces, in that there is a morphism into it from an adic space built out of the rings $\tilde{\mathcal{R}}_{L,A}^{[s,r]}$, the pullback along which induces an equivalence of categories of coherent sheaves by analogy with Serre’s GAGA theorem in complex algebraic geometry. This adic space in turn admits an infinite cyclic étale cover which is a “quasi-Stein space” whose global sections are the ring $\bigcap_{r>0} \tilde{\mathcal{R}}_{L,A}^r$, on which the deck transformations act via the powers of φ . For more on this story, see the aforementioned references such as [20], and also [36, §8.7, 8.8] and [37, §4.7].

Remark 2.20. Let F/K be an algebraic extension. In the case where $F = \mathbb{Q}_p(\mu_{p^\infty})$, we have seen already (Example 2.2) that the completion of F is a perfectoid field. This property turns to be closely related to ramification of local fields; let us now recall the precise nature of this relationship.

Coates–Greenberg [15] define F/K to be *deeply ramified* if for every finite extension F' of F , the trace map $\mathrm{Trace} : \mathfrak{m}_{F'} \rightarrow \mathfrak{m}_F$ is surjective. This holds in particular if F/K is *arithmetically profinite* in the sense of Fontaine–Wintenberger [22] (see [21, Corollary 1.5] for a detailed proof); the latter holds in turn if F/K is an infinite Galois extension with finite residual extension whose Galois group is a p -adic Lie group, by a theorem of Sen [49].

Lemma 2.21. *Let F/K be an algebraic extension with completion L . Then F/K is deeply ramified if and only if L is a perfectoid field.*

Proof. Suppose first that L is not perfectoid. If the value group of L is not p -divisible or the residue field of L is not perfect, it is obvious that F/K is not discretely ramified. Otherwise, there exists some $\lambda \in \mathfrak{m}_F$ such that λ^p divides p in \mathfrak{o}_F and the Frobenius map from $\mathfrak{o}_F/(\lambda)$ to $\mathfrak{o}_F/(\lambda^p)$ must fail to be surjective. Choose $x \in \mathfrak{o}_F$ whose image in $\mathfrak{o}_F/(\lambda^p)$ is not in the image of Frobenius. Since the set of p -th powers in \mathfrak{o}_F is closed, the minimum of $|x - y^p|$ over all $y \in \mathfrak{o}_F$ is achieved by some element y_0 . Since by hypothesis $|x - y_0^p| > |\lambda^p|$, we may choose $\mu \in L$ with $|\mu^p| = |x - y_0^p|$ and write the integral closure of \mathfrak{o}_F in $F(x^{1/p})$ as the set of elements of the form

$$\sum_{i=0}^{p-1} z_i \left(\frac{x^{1/p} - y_0}{\mu} \right)^i \quad (z_0, \dots, z_{p-1} \in \mathfrak{o}_F).$$

The trace of such an element is $\sum_{i=0}^{p-1} p z_i (-y_0/\mu)^i$, which is necessarily divisible by p/μ^{p-1} ; hence F/K is not deeply ramified.

Conversely, suppose that L is perfectoid. For any finite extension F' of F , by Theorem 2.4 the completion L' of F' is again perfectoid. Using the existence of a commutative diagram

$$\begin{array}{ccc} W(\mathfrak{o}_{L^b}) & \longrightarrow & \mathfrak{o}_L \\ \downarrow & & \downarrow \\ W(\mathfrak{o}_{L'^b}) & \longrightarrow & \mathfrak{o}_{L'} \end{array}$$

in which the horizontal arrows are surjective (see Definition 2.14), the surjectivity of $\text{Trace} : \mathfrak{m}_{F'} \rightarrow \mathfrak{m}_F$ reduces to the surjectivity of $\text{Trace} : \mathfrak{m}_{L'^b} \rightarrow \mathfrak{m}_{L^b}$, or equivalently the fact that the cokernel of the latter map is annihilated by all of \mathfrak{m}_{L^b} . This holds because the annihilator of the cokernel is nonzero (because L^b/L^b is a finite separable extension) and closed under taking p -th roots. (This argument is a special case of the *almost purity theorem* for perfectoid rings; see [36, Theorem 5.5.9] and [46, Theorem 7.9].) \square

3. SLOPES OF φ -MODULES

We now introduce the important concept of *slopes* of φ -modules. The basic theory is motivated by the corresponding theory of slopes of vector bundles on algebraic varieties (especially curves). In the process, we identify the essential image of the embedding functor of Theorem 2.13 in case A is a field.

Hypothesis 3.1. Throughout §3, let L be a perfectoid field.

Lemma 3.2. *Suppose that A is a field. Then $\tilde{\mathcal{R}}_{L,A}^\times = \bigcup_{r>0} (W^r(L)[p^{-1}] \hat{\otimes}_{\mathbb{Q}_p} A)^\times$. (This statement can be extended to the case where A is reduced, but not more generally.)*

Proof. See [36, Corollary 4.2.5]. \square

Definition 3.3. Suppose that A is a field. Let $v(A)$ denote the valuation group of A , normalized so that $v(\mathbb{Q}_p^\times) = \mathbb{Z}$. Let k be the largest finite extension of \mathbb{F}_p which embeds into both L and the residue field of A , and let A_0 be the unramified extension of \mathbb{Q}_p with residue field k ; then $W(L) \hat{\otimes}_{\mathbb{Q}_p} A = W(L) \otimes_{\mathbb{Q}_p} A$ splits into copies of the integral domain $W(L) \otimes_{A_0} A$ indexed by choices of the embedding $k \rightarrow L$. The p -adic valuation on $W(L)$ extends to a valuation $W(L) \otimes_{A_0} A$ with values in $v(A)$; summing across components gives a map $W(L) \otimes_{\mathbb{Q}_p} A \rightarrow v(A) \cup \{+\infty\}$. By Lemma 3.2, we obtain a homomorphism $\tilde{\mathcal{R}}_{L,A}^\times \rightarrow v(A)$; note that this map is invariant under φ -pullback.

By Theorem 2.18, line bundles on $X_{L,A}$ correspond to φ -modules over $\tilde{\mathcal{R}}_{L,A}$ whose underlying modules are projective of rank 1. By taking determinants of these modules and using the φ -invariance of the map $\tilde{\mathcal{R}}_{L,A}^\times \rightarrow v(A)$, we obtain a morphism $\deg : \text{Pic}(X_{L,A}) \rightarrow v(A)$ called the *degree map*. (This map can also be given an interpretation in terms of rational sections of line bundles, in parallel with the usual construction of the degree map for line bundles on an algebraic curve; see [20] for this viewpoint.) As usual, for $\mathcal{F} \in \mathbf{VB}_{L,A}$ of arbitrary rank, we define the *degree* of \mathcal{F} as the degree of its determinant $\wedge^{\text{rank}(\mathcal{F})} \mathcal{F}$, and (if $\mathcal{F} \neq 0$) the *slope* of \mathcal{F} as the ratio $\mu(\mathcal{F}) = \deg(\mathcal{F})/\text{rank}(\mathcal{F})$. We may transfer these definitions to $\Phi\mathbf{Mod}_{L,A}$ using Theorem 2.18.

Lemma 3.4. *For each positive integer n , we have $\ker(p^n \varphi - 1 : \tilde{\mathcal{R}}_{L,A}^{[s,r]} \rightarrow \tilde{\mathcal{R}}_{L,A}^{[s,r/p]}) = 0$.*

Proof. Again using [36, Lemma 2.2.9(b)], we reduce to the case $A = \mathbb{Q}_p$. Suppose that x belongs to the kernel. The equality

$$|x|_t = |\varphi(x)|_{t/p} = |p^{-n}x|_{t/p} = p^n |x|_{t/p}$$

holds initially for $t \in [s, r]$, then by induction for $t \in [p^{-m}s, r]$ for each nonnegative integer m , and hence for all $t \in (0, r]$. It follows that $|x|_t$ remains bounded as $t \rightarrow 0^+$, so by [36, Lemma 4.2.4] we have $x \in \tilde{\mathcal{R}}_{L,A}^\times$. However, by Lemma 3.2, this means that x has a well-defined p -adic valuation, which is the same as the valuation of $\varphi(x)$; we must then have $n = 0$, contradiction. \square

Definition 3.5. Suppose that A is a field. For $\mathcal{F} \in \mathbf{VB}_{L,A}$ nonzero, we say \mathcal{F} is *stable* (resp. *semistable*) if there does not exist a nonzero proper subobject \mathcal{F}' of \mathcal{F} such that $\mu(\mathcal{F}') \geq \mu(\mathcal{F})$ (resp. $\mu(\mathcal{F}') > \mu(\mathcal{F})$). For example, by Lemma 3.4, any rank 1 bundle is semistable. We say \mathcal{F} is *étale* if it is semistable of degree 0.

For general $\mathcal{F} \in \mathbf{VB}_{L,A}$, Lemma 3.4 implies that the set of slopes of nonzero subbundles of \mathcal{F} is bounded above; consequently, there exists a canonical filtration

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \cdots \subset \mathcal{F}_m = \mathcal{F}$$

such that the successive quotients $\mathcal{F}_i/\mathcal{F}_{i-1}$ are stable and $\mu(\mathcal{F}_1/\mathcal{F}_0) > \cdots > \mu(\mathcal{F}_m/\mathcal{F}_{m-1})$. This filtration is called the *Harder-Narasimhan filtration*, or *HN filtration*, of \mathcal{F} . Note that \mathcal{F}_1 is the maximal subbundle of \mathcal{F} achieving the maximal slope among nonzero subbundles of \mathcal{F} . The *HN polygon* of \mathcal{F} is the Newton polygon of length $\text{rank}(\mathcal{F})$ in which the slope $\mu(\mathcal{F}_i/\mathcal{F}_{i-1})$ occurs with multiplicity $\text{rank}(\mathcal{F}_i)$; the total height of this polygon is $\deg(\mathcal{F})$.

Theorem 3.6. *Suppose that A is a field.*

- (a) *The tensor product of any two semistable bundles in $\mathbf{VB}_{L,A}$ is again semistable. In particular, the tensor product of two étale bundles is again étale.*
- (b) *Let L' be any perfectoid field containing L . Then the HN polygon remains invariant under base extension from $\mathbf{VB}_{L,A}$ to $\mathbf{VB}_{L',A}$.*
- (c) *The essential image of the full embedding $\mathbf{Rep}_A(G_L) \rightarrow \Phi\mathbf{Mod}_{L,A}$ in Theorem 2.13 consists precisely of the étale objects.*

Proof. See [36, §4] and references therein. \square

Remark 3.7. For general A , we may define the degree, rank, slope, and HN polygon of $\mathcal{F} \in \mathbf{VB}_{L,A}$ as functions on $\text{Maxspec}(A)$. Unfortunately, these functions do not extend well to the Berkovich space associated to A , because the theory of slopes behaves poorly when the degree map does not take discrete values.

In addition, if A is not a field, then the subcategory of $\Phi\mathbf{Mod}_{L,A}$ consisting of pointwise étale objects may be strictly larger than the essential image of $\mathbf{Rep}_A(G_L)$; in fact, this already occurs for objects of rank 1, as noted in [6, Remarque 4.2.10]. For further discussion, see [35].

Lemma 3.8. *For $\mathcal{F} \in \mathbf{VB}_{L,A}$, the HN polygon of \mathcal{F} , as a function on $\text{Maxspec}(A)$, is bounded above and below, and its height is constant on connected components of $\text{Maxspec}(A)$.*

Proof. Using Lemma 2.12, we see that \mathcal{F} admits a surjective morphism from $\mathcal{O}(n)^{\oplus d}$ for some integers n, d . It follows that the HN polygon of \mathcal{F} has no slopes less than n . The same argument applies to the dual bundle shows that the HN polygon of \mathcal{F} also has slopes which are uniformly bounded above. \square

4. FROM φ -MODULES TO (φ, Γ) -MODULES

We now use φ -modules to give an alternate description of the category $\Phi\Gamma_{K,A}$ in the language of perfectoid fields. In the process, we will establish Theorem 1.7.

Definition 4.1. Let L_K be the completion of $K(\mu_{p^\infty})$; it is a perfectoid field by Example 2.2 and Theorem 2.4. Let $\widetilde{\Phi\Gamma}_{K,A}$ denote the category of objects of $\Phi\mathbf{Mod}_{L_K,A}$ equipped with continuous semilinear Γ_K -actions, where $\Gamma_K = \mathrm{Gal}(K(\mu_{p^\infty})/K)$.

We now complete the discussion initiated in Remark 1.9.

Remark 4.2. Via the cyclotomic character, we may identify Γ_F with Γ and Γ_K with an open subgroup of Γ . Put $\tilde{\mathcal{R}}_{K,A} = \mathrm{Ind}_{\Gamma_K}^\Gamma \tilde{\mathcal{R}}_{L_K,A}$; this is a direct sum of copies of $\tilde{\mathcal{R}}_{L_K,A}$ indexed by the connected components of $K \otimes_F F(\mu_{p^\infty})$. We may then identify objects of $\widetilde{\Phi\Gamma}_{K,A}$ with finite projective $\tilde{\mathcal{R}}_{K,A}$ -modules equipped with continuous semilinear Γ -actions.

Definition 4.3. Choose a coherent sequence $\zeta_p, \zeta_{p^2}, \dots$ of p -power roots of unity and let ϵ be the element $(1, \zeta_p, \zeta_{p^2}, \dots) \in L_K^\flat$. Then the map $W(k)[[\pi]] \rightarrow W(\mathfrak{o}_{L_K^\flat})$ taking π to $[\epsilon] - 1$ is (φ, Γ) -equivariant; it thus extends to a (φ, Γ) -equivariant map $\mathcal{R}_{K,A} \rightarrow \tilde{\mathcal{R}}_{K,A}$.

Theorem 4.4. *The categories $\Phi\Gamma_{K,A}$ and $\widetilde{\Phi\Gamma}_{K,A}$ are equivalent via base extension along $\mathcal{R}_{K,A} \rightarrow \tilde{\mathcal{R}}_{K,A}$.*

Proof. In the case $A = \mathbb{Q}_p$, this is proved in [37, Theorem 6.1.7]. It can also be deduced from prior results; for example, for L a completed algebraic closure of K , Berger [5, Théorème A] constructed an equivalence between $\Phi\Gamma_{K,A}$ and the category of objects of $\mathbf{BPair}_{L,A}$ equipped with continuous semilinear G_K -actions. By Theorem 2.18, these can be interpreted as objects of $\Phi\mathbf{Mod}_{L,A}$ equipped with continuous semilinear G_K -actions; using Lemma 2.6 and Lemma 2.10, these can in turn be identified with objects of $\widetilde{\Phi\Gamma}_{K,A}$.

To obtain full faithfulness in the general case, note that since both categories admit internal Homs in a compatible way, we reduce to checking that for $M \in \Phi\Gamma_{K,A}$, every (φ, Γ) -stable element $\mathbf{v} \in M \otimes_{\mathcal{R}_{K,A}} \tilde{\mathcal{R}}_{K,A}$ belongs to M itself. Using a Schauder basis for A over \mathbb{Q}_p (see the proof of Lemma 2.6), we may construct a family of bounded \mathbb{Q}_p -linear morphisms $A \rightarrow \mathbb{Q}_p$ whose kernels have zero intersection; by tensoring along these, we reduce the claim that $\mathbf{v} \in M$ to a family of corresponding assertions in the previously treated case $A = \mathbb{Q}_p$.

To obtain essential surjectivity, one may emulate the proof of [37, Theorem 6.1.7]; we only give a brief sketch here, as details will be given in upcoming work of Chojecki and Gaisin. By full faithfulness, we may reduce to the case $K = F$. By Lemma 2.10, we may start with a finite projective $\tilde{\mathcal{R}}_{L_F,A}^{[s,r]}$ -module \tilde{M} equipped with an isomorphism of the base extensions of $\varphi^* \tilde{M}$ and \tilde{M} to $\tilde{\mathcal{R}}_{L_F,A}^{[s,r/p]}$, plus a compatible semilinear Γ -action. Within $\tilde{\mathcal{R}}_{L_F,A}^{[s,r]}$, we have a dense subring $\check{\mathcal{R}}_{F,A}^{[s,r]}$ consisting of the union of the closures of the subrings $\varphi^{-n}((F \hat{\otimes}_{\mathbb{Q}_p} A)[\pi^\pm])$ for $n \geq 0$; it will suffice to descend \tilde{M} to a finite projective $\check{\mathcal{R}}_{F,A}^{[s,r]}$ -module \check{M} on which φ and Γ continue to act. Using the density of $\check{\mathcal{R}}_{F,A}^{[s,r]}$ in $\tilde{\mathcal{R}}_{L_F,A}^{[s,r]}$, we may apply [37, Lemma 5.6.8] to descend the underlying module of \tilde{M} , but the resulting descended module will typically not be Γ -stable; this may be corrected using a sequence of successive approximations as in [37, Lemma 5.6.9]. We thus obtain a Γ -stable descended module, which is then easily shown to be also φ -stable. \square

As a corollary, we may now establish Theorem 1.5.

Proof of Theorem 1.5. By Theorem 4.4, it suffices to exhibit a full embedding $\mathbf{Rep}_A(G_K) \rightarrow \widetilde{\Phi\Gamma}_{K,A}$. We obtain this embedding from Theorem 2.13 by adding Γ_K -descent data. \square

Definition 4.5. We say that $M \in \Phi\Gamma_{K,A}$ is *étale* if its image in $\Phi\mathbf{Mod}_{L_K,A}$ is étale.

Theorem 4.6. Suppose that A is a field. Then $M \in \Phi\Gamma_{K,A}$ is étale if and only if it belongs to the essential image of the functor $\mathbf{Rep}_A(G_K) \rightarrow \Phi\Gamma_{K,A}$.

Proof. This is immediate from Theorem 3.6(c). \square

Remark 4.7. Theorem 4.4, when restricted to étale objects, reproduces the Cherbonnier-Colmez theorem on the overconvergence of p -adic representations [14]. However, the proof we have in mind is closer in spirit to the one in [32, §2].

Remark 4.8. Note that the embedding in Theorem 1.5 is not quite canonical: it depends on the coherent sequence of p -power roots of unity chosen in Definition 4.3. This suggests that in some sense, the embedding of $\mathbf{Rep}_{G_K}(A)$ into $\widetilde{\Phi\Gamma}_{K,A}$ is more natural than the embedding into $\Phi\Gamma_{K,A}$.

Remark 4.9. Let F/K be a deeply ramified Galois algebraic extension, so that by Lemma 2.21 the completion L of F is a perfectoid field. Using Lemma 2.6, we may describe the category $\widetilde{\Phi\Gamma}_{K,A}$ as the category of objects of $\Phi\mathbf{Mod}_{L,A}$ equipped with continuous semilinear $\mathrm{Gal}(F/K)$ -actions; in particular, we again obtain a full embedding of $\mathbf{Rep}_A(G_K)$ into this category. However, in general there is no natural analogue of the category $\Phi\Gamma_{K,A}$ because the ring $\tilde{\mathcal{R}}_{L,A}$ cannot be obtained in a natural way from a ring of Laurent series. For this reason, we are driven to reformulate known constructions involving $\Phi\Gamma_{K,A}$ using $\widetilde{\Phi\Gamma}_{K,A}$ with the eye towards generalizing to towers other than the cyclotomic tower; we will pick up on this theme in §8.

5. COHOMOLOGY OF (φ, Γ) -MODULES

We now upgrade the previous discussion to relate Galois cohomology to (φ, Γ) -modules. This time, we start directly with the perfectoid framework.

Definition 5.1. Let L be a perfectoid field. For $\tilde{M} \in \Phi\mathbf{Mod}_{L,A}$, let $H_\varphi^0(\tilde{M}), H_\varphi^1(\tilde{M})$ be the kernel and cokernel of $\varphi - 1$ on \tilde{M} , and put $H_\varphi^i(\tilde{M}) = 0$ for $i > 0$.

Lemma 5.2. Choose r, s with $0 < s \leq r/p$. Let $\tilde{M}, \tilde{M}^r, \tilde{M}^{[s,r]}$ be corresponding objects in the categories (a), (b), (c) of Lemma 2.10. Then the vertical arrows in the diagram

$$\begin{array}{ccccccc}
0 & \longrightarrow & \tilde{M} & \xrightarrow{\varphi-1} & \tilde{M} & \longrightarrow & 0 \\
& & \uparrow & & \uparrow & & \\
0 & \longrightarrow & \tilde{M}^r & \xrightarrow{\varphi-1} & \tilde{M}^r \otimes_{\tilde{\mathcal{R}}_{L,A}^r} \tilde{\mathcal{R}}_{L,A}^{r/p} & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \tilde{M}^{[s,r]} & \xrightarrow{\varphi-1} & \tilde{M}^{[s,r]} \otimes_{\tilde{\mathcal{R}}_{L,A}^{[s,r]}} \tilde{\mathcal{R}}_{L,A}^{[s,r/p]} & \longrightarrow & 0
\end{array}$$

constitute quasi-isomorphisms of the horizontal complexes.

Proof. The proof of [36, Proposition 6.3.19] in the case $A = \mathbb{Q}_p$ adapts without change. \square

Definition 5.3. For G a profinite group acting continuously on a topological abelian group M , let $C(G, M)$ denote the complex of inhomogeneous continuous cochains on G with values in M . Denote by $H^i(G, M)$ or $H_G^i(M)$ the cohomology groups of this complex.

For $M \in \Phi\Gamma_{K,A}$, let $C_{\varphi,\Gamma}(M)$ denote the total complex associated to the double complex

$$0 \rightarrow C(\Gamma, M) \xrightarrow{\varphi^{-1}} C(\Gamma, M) \rightarrow 0.$$

Denote by $H_{\varphi,\Gamma}^i(M)$ the cohomology groups of this complex. We make an analogous definition for $\tilde{M} \in \widetilde{\Phi\Gamma}_{K,A}$.

Lemma 5.4. *Let L be a perfectoid field. Let L' be the completion of a (possibly infinite) Galois algebraic extension of L with Galois group G . Then for $0 < s \leq r$, for M a finite projective $\check{\mathcal{R}}_{L,A}^{[s,r]}$ -module, M is G -acyclic, i.e., the morphism $M \rightarrow C(G, M)$ is a quasi-isomorphism.*

Proof. Using a Schauder basis for A over \mathbb{Q}_p (see the proof of Lemma 2.6), we may reduce to the case $A = \mathbb{Q}_p$, for which we may apply [36, Theorem 8.2.22]. \square

Theorem 5.5. *Let L be a perfectoid field. For $\tilde{M} \in \Phi\mathbf{Mod}_{L,A}$, $\mathcal{F} \in \mathbf{VB}_{L,A}$ corresponding as in Theorem 2.18, we have canonical identifications $H_{\varphi}^i(\tilde{M}) \cong H^i(X_{L,A}, \mathcal{F})$ for all $i \geq 0$.*

Proof. Since $X_{L,A}$ is separated and is covered by two open affine subschemes, we have $H^i(X_{L,A}, \mathcal{F}) = 0$ for $i \geq 2$. The identification for $i = 0, 1$ arises directly from Theorem 2.18, using in the case $i = 1$ the interpretation of the cohomology groups as Yoneda extension groups. \square

Theorem 5.6. *For $M \in \Phi\Gamma_{K,A}$, $\tilde{M} \in \widetilde{\Phi\Gamma}_{K,A}$ corresponding via Theorem 4.4, the morphisms $H_{\varphi,\Gamma}^i(M) \rightarrow H_{\varphi,\Gamma}^i(\tilde{M})$ are isomorphisms for all i .*

Proof. As in Theorem 4.4, the case $A = \mathbb{Q}_p$ is treated in [37, Theorem 6.1.7], and we sketch an adaptation to the general case and refer to upcoming work of Chojecki and Gaisin for further details. We again reduce to the case $K = F$, and to calculating (φ, Γ) -cohomology for a pair of modules \check{M}, \tilde{M} in which \check{M} is finite projective over $\check{\mathcal{R}}_{F,A}^{[s,r]}$ and \tilde{M} is the base extension to $\check{\mathcal{R}}_{L,F,A}^{[s,r]}$. In this setting, we may already show that $H_{\Gamma}^i(\check{M}) = H_{\Gamma}^i(\tilde{M})$ using the method of [37, Lemma 5.6.6], i.e., by first making a direct calculation in the case where $\check{M} = \check{\mathcal{R}}_{F,A}^{[s,r]}$, then using this case to make a series of successive approximations in the general case. \square

Theorem 5.7. *Suppose that $[K : \mathbb{Q}_p] < \infty$. For $M \in \Phi\Gamma_{K,A}$, we have the following.*

- (a) *The groups $H_{\varphi,\Gamma}^i(M)$ are finite A -modules for $i = 0, 1, 2$, and vanish for $i > 2$.*
- (b) *For any morphism $A \rightarrow B$ of affinoid algebras over \mathbb{Q}_p , the canonical morphism*

$$C_{\varphi,\Gamma}(M) \otimes_A^{\mathbb{L}} B \rightarrow C_{\varphi,\Gamma}(M \otimes_{\mathcal{R}_{K,A}} \mathcal{R}_{K,B})$$

is a quasi-isomorphism.

- (c) *If M is the image of $V \in \mathbf{Rep}_A(G_K)$ under Theorem 1.5, then there is a canonical quasi-isomorphism $C(G_K, V) \cong C_{\varphi,\Gamma}(M)$. In particular, the A -modules $H_{\varphi,\Gamma}^i(M)$ coincide with the Galois cohomology groups of V .*

Proof. See [39, Proposition 2.3.7, Theorem 4.4.2, Theorem 4.4.3]. \square

Remark 5.8. While Theorem 5.6 and Theorem 5.7 together assert that the groups $H_{\varphi, \Gamma}^i(\tilde{M})$ are finite A -modules, the proof of this statement depends crucially on the interpretation of \tilde{M} in terms of the category $\Phi\Gamma_{K,A}$. To illustrate this, we sketch a proof of Theorem 5.7(a) in the spirit of [39] but technically somewhat simpler. (It is also slightly more general, as we only need to assume that A is a noetherian Banach algebra over \mathbb{Q}_p .)

We first reduce to the case $K = \mathbb{Q}_p$ using the (φ, Γ) -module-theoretic counterpart of Shapiro's lemma described in [41, Theorem 3.2] (compare also Remark 4.2). For $M \in \Phi\Gamma_{A,K}$, Shapiro's lemma as usual implies that $H_{\varphi, \Gamma_K}^i(M) \cong H_{\varphi, \Gamma}^i(\text{Ind}_{\Gamma_K}^{\Gamma} M)$; since the definition of the latter does not explicitly reference the module structure, we may view $\text{Ind}_{\Gamma_K}^{\Gamma} M$ as a module over $\text{Ind}_{\Gamma_K}^{\Gamma} \mathcal{R}_{K,A}$ and then restrict scalars to $\mathcal{R}_{\mathbb{Q}_p,A}$ without changing the cohomology.

Now assuming $K = \mathbb{Q}_p$, choose r, s, r', s' with $0 < s < s' \leq r'/p \leq r/p$. Let $\mathcal{R}_{\mathbb{Q}_p,A}^{[s,r]}$ be the ring of rigid analytic functions on the disc $p^{-rp/(p-1)} < |\pi| < p^{-sp/(p-1)}$ over A . For r sufficiently small, we may (by analogy with Lemma 2.10 and Lemma 5.2) realize M as a finite projective $\mathcal{R}_{\mathbb{Q}_p,A}^{[s,r]}$ -module $M^{[s,r]}$ equipped with an isomorphism

$$\varphi^* M^{[s,r]} \otimes_{\mathcal{R}_{\mathbb{Q}_p,A}^{[s/p, r/p]}} \mathcal{R}_{\mathbb{Q}_p,A}^{[s, r/p]} \cong M^{[s,r]} \otimes_{\mathcal{R}_{\mathbb{Q}_p,A}^{[s,r]}} \mathcal{R}_{\mathbb{Q}_p,A}^{[s, r/p]}$$

and compute $H_{\varphi, \Gamma}^i(M)$ as the cohomology of the total complex

$$0 \rightarrow C(\Gamma, M^{[s,r]}) \xrightarrow{\varphi-1} C(\Gamma, M^{[s, r/p]}) \rightarrow 0.$$

We then have a diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C(\Gamma, M^{[s,r]}) & \xrightarrow{\varphi-1} & C(\Gamma, M^{[s, r/p]}) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & C(\Gamma, M^{[s', r']}) & \xrightarrow{\varphi-1} & C(\Gamma, M^{[s', r'/p]}) & \longrightarrow & 0 \end{array}$$

in which the vertical arrows define a quasi-isomorphism of the total complexes associated to the rows. However, each vertical arrow is composed of maps which are *completely continuous* morphisms of Banach spaces over A , i.e., uniform limits of morphisms of finite rank. By the Cartan-Serre-Schwartz lemma as applied in [38, §3] (compare [40, Satz 2.6]), we deduce that the cohomology groups of the total complexes are finite A -modules. (By contrast, the maps $\tilde{\mathcal{R}}_{\mathbb{Q}_p,A}^{[s,r]} \rightarrow \tilde{\mathcal{R}}_{\mathbb{Q}_p,A}^{[s', r']}$ are not completely continuous.)

Remark 5.9. In Remark 5.8, note that φ is only A -linear rather than $(F \hat{\otimes}_{\mathbb{Q}_p} A)$ -linear; we thus need $[K : \mathbb{Q}_p] < \infty$ in order to reduce to the case $K = F = \mathbb{Q}_p$. If we relax the hypothesis on K to allow it to be a more general local field, the vanishing of $H_{\varphi, \Gamma}^i(M)$ for $i = 0$ and $i > 2$ remains valid, but the finite generation of $H_{\varphi, \Gamma}^i$ for $i = 1$ and $i = 2$ can fail.

6. THE CYCLOTOMIC DEFORMATION

We now consider a key example of an arithmetic deformation. The construction follows [39, Definition 4.4.7], but we opt here for more geometric language.

Definition 6.1. Let X be a rigid analytic space over \mathbb{Q}_p . The rings $\mathcal{R}_{K,A}, \tilde{\mathcal{R}}_{K,A}^{[s,r]}, \tilde{\mathcal{R}}_{K,A}^r, \tilde{\mathcal{R}}_{K,A}$ all satisfy the sheaf axiom and Tate acyclicity with respect to finite coverings by affinoid subdomains: for example, the Čech sequence for $\tilde{\mathcal{R}}_{K,A}$ with respect to a given covering is obtained from the corresponding sequence for the structure sheaf by the exact operation of taking the completed tensor product over \mathbb{Q}_p with $\tilde{\mathcal{R}}_{K,\mathbb{Q}_p}$. (The exactness of completed tensor products over \mathbb{Q}_p does involve a nontrivial argument using Schauder bases; see for example [36, Lemma 2.2.9].) These constructions thus give rise to ring sheaves $\mathcal{R}_{K,X}, \tilde{\mathcal{R}}_{K,X}^{[s,r]}, \tilde{\mathcal{R}}_{K,X}^r, \tilde{\mathcal{R}}_{K,X}$ on the affinoid space $\text{Maxspec}(A)$ which are acyclic on affinoid subspaces.

Let $\Phi\Gamma_{K,X}, \widetilde{\Phi\Gamma}_{K,X}$ be the categories of finite projective modules over the respective ring sheaves $\mathcal{R}_{K,X}, \tilde{\mathcal{R}}_{K,X}$ equipped with continuous commuting semilinear actions of φ, Γ . These form stacks for both the analytic topology and the étale topology; in particular, Theorem 4.4 gives rise to an equivalence of categories $\Phi\Gamma_{K,X} \rightarrow \widetilde{\Phi\Gamma}_{K,X}$.

Definition 6.2. Let $\mathbb{Z}_p[[\Gamma_K]]$ be the completed group algebra. Since this ring is formally of finite type over \mathbb{Z}_p , we may apply Berthelot's generic fiber construction (see for example [17, §7]) to view this ring as the collection of bounded-by-1 rigid analytic functions on a certain one-dimensional quasi-Stein space W_K over \mathbb{Q}_p (the *weight space* of Γ_K). More precisely, in case $\Gamma_K \cong \mathbb{Z}_p$, the space W_K is an open unit disc admitting $\gamma - 1$ as a coordinate for any topological generator $\gamma \in \Gamma_K$; in the general case, W_K is a finite disjoint union of such discs.

The action of $\mathbb{Z}_p[[\Gamma_K]]$ on $\Lambda_K = \mathcal{O}(W_K)$ by (left) multiplication defines a canonical one-dimensional Galois representation on W_K ; let \mathbf{Dfm}_K be the corresponding (φ, Γ_K) -module. For X a rigid analytic space over \mathbb{Q}_p and $M \in \Phi\Gamma_{K,X}$, define the *cyclotomic deformation* of M as the external tensor product $M \boxtimes \mathbf{Dfm}_K \in \Phi\Gamma_{K,X \times_{\mathbb{Q}_p} W_K}$. We similarly define the cyclotomic deformation of $\tilde{M} \in \widetilde{\Phi\Gamma}_{K,X}$ as an object $\tilde{M} \boxtimes \mathbf{Dfm}_K$ of $\widetilde{\Phi\Gamma}_{K,X \times_{\mathbb{Q}_p} W_K}$.

Remark 6.3. For $M \in \Phi\Gamma_{K,A}$, we may view the cyclotomic deformation of M as arising from the completed tensor product $M' = M \hat{\otimes}_{\mathbb{Q}_p} \Lambda_K$. Since the action of Γ_K on M is locally analytic, the action map $M \times \Gamma_K \rightarrow M$ extends to a projection $M' \rightarrow M$ sectioning the inclusion $M \rightarrow M'$. From the existence of this projection, it follows (as in the proof of [39, Theorem 4.4.8]) that

$$(M')^{\Gamma_K} = 0, \quad (M')_{\Gamma_K} \cong M.$$

By contrast, for $\tilde{M} \in \widetilde{\Phi\Gamma}_{K,A}$, the action map $\tilde{M} \times \Gamma_K \rightarrow \tilde{M}$ is continuous, but does not define a bounded map $\tilde{M} \hat{\otimes}_{\mathbb{Q}_p} \Lambda_K \rightarrow \tilde{M}$.

Definition 6.4. Define the ring $\check{\mathcal{R}}_{K,A} = \bigcup_{n=0}^{\infty} \varphi^{-n}(\mathcal{R}_{K,A})$. We may then define the category $\check{\Phi\Gamma}_{K,A}$ of (φ, Γ) -modules over $\check{\mathcal{R}}_{K,A}$. The base extension functor $\Phi\Gamma_{K,A} \rightarrow \check{\Phi\Gamma}_{K,A}$ is obviously surjective; by Theorem 4.4 it is also fully faithful, and hence an equivalence.

7. IWASAWA COHOMOLOGY AND THE CYCLOTOMIC DEFORMATION

The goal of this section is to describe various constructions in the classical language of (φ, Γ) -modules which play a role in Iwasawa theory, then translate these into the other categories so as to isolate the role of the cyclotomic extension. Assume hereafter that $[K : \mathbb{Q}_p] < \infty$.

Definition 7.1. Define the map $\psi : \mathcal{R}_{K,A} \rightarrow \mathcal{R}_{K,A}$ as the reduced trace of φ , i.e., as $p^{-1}\varphi^{-1} \circ \text{Trace}(\mathcal{R}_{K,A} \rightarrow \varphi(\mathcal{R}_{K,A}))$; by definition, it is a left inverse of φ . For any $M \in \Phi\Gamma_{K,A}$, we may likewise take the reduced trace of the action of φ on M to obtain an action of ψ on M , which is again a left inverse of φ ; concretely, the action of ψ on M is characterized by additivity and the identity

$$\psi(r\varphi(\mathbf{v})) = \psi(r)\mathbf{v}.$$

We have an exact sequence

$$(7.1.1) \quad 0 \rightarrow M^{\varphi=1} \rightarrow M^{\psi=1} \xrightarrow{\varphi^{-1}} M^{\psi=0}.$$

Proposition 7.2. *For any $M \in \Phi\Gamma_{K,A}$, the A -module $M/(\psi - 1)$ is finitely generated.*

Proof. It suffices to treat the case $K = \mathbb{Q}_p$, for which see [39, Proposition 3.3.2(1)]. \square

Corollary 7.3. *For any $M \in \Phi\Gamma_{K,A}$, there is a canonical isomorphism*

$$M^{\varphi=1} \cong \text{Hom}_A(M^*/(\psi - 1), A).$$

In particular, the A -module $M^{\varphi=1}$ is finitely generated.

Proof. It suffices to treat the case $K = \mathbb{Q}_p$. In this case, using residues of power series, one constructs as in [39, Notation 2.3.13] a nondegenerate pairing $M \times M^* \rightarrow \mathcal{R}_{K,A}(1)$ satisfying

$$\{\varphi(x), \varphi(y)\} = \{x, y\} \quad (x \in M, y \in M^*)$$

and hence

$$(7.3.1) \quad \{\varphi(x), y\} = \{x, \psi(y)\} \quad (x \in M, y \in M^*).$$

We will show that the map $M \rightarrow \text{Hom}_A(M^*, A)$ arising from the pairing $\{-, -\}$ induces the desired isomorphism; this will then imply the finite generation of $M^{\varphi=1}$ using Proposition 7.2.

To begin with, the nondegeneracy of the pairing $\{-, -\}$ implies the injectivity of $M \rightarrow \text{Hom}_A(M^*, A)$, and the identity (7.3.1) shows that the image of this map is contained in $\text{Hom}_A(M^*/(\psi - 1), A)$. In the other direction, note that Proposition 7.2 and the open mapping theorem imply that $(\psi - 1)M^*$ is a closed subspace of M^* for the Fréchet topology, so every element of $\text{Hom}_A(M^*/(\psi - 1), A)$ defines a continuous A -linear map $M^* \rightarrow A$. By the perfectness of the pairing, any such map corresponds to an element of M . \square

Remark 7.4. One of the key constructions in p -adic Hodge theory is Fontaine's *crystalline period functor*, which takes a representation $V \in \mathbf{Rep}_A(G_K)$ to the finite A -module

$$D_{\text{crys}}(V) = (V \otimes_{\mathbb{Q}_p} \mathbf{B}_{\text{crys}})^{G_K}$$

where \mathbf{B}_{crys} is a certain topological \mathbb{Q}_p -algebra (the *ring of crystalline periods*). For example, when $A = \mathbb{Q}_p$ and V is the p -adic étale cohomology of a smooth proper scheme over \mathfrak{o}_K , the *crystalline comparison theorem* defines a functorial isomorphism of $D_{\text{crys}}(V)$ with the rational crystalline cohomology of the same scheme; see [9] for a thoroughly modern take on the construction.

The functor D_{crys} factors naturally through $\Phi\Gamma_{K,A}$ and $\widetilde{\Phi\Gamma}_{K,A}$: for example, for $M = \Phi\Gamma_{K,A}$, $\tilde{M} \in \widetilde{\Phi\Gamma}_{K,A}$ corresponding to $V \in \mathbf{Rep}_A(G_K)$, we have a canonical isomorphism

$$D_{\text{crys}}(V) \cong M[t^{-1}]^\Gamma \cong \tilde{M}[t^{-1}]^\Gamma, \quad t = \log(1 + \pi).$$

In the case of \tilde{M} , all we are using about t is that it belongs to $P_{L_K, A, n}$ for some $n > 0$ and that its zero locus on $X_{L, A}$ is precisely $Z_{L, A}$; this interpretation can be used to avoid specific references to the cyclotomic tower.

When K/\mathbb{Q}_p is unramified, A is finite over \mathbb{Q}_p , and $M \in \Phi\Gamma_{K, A}$ is crystalline (i.e., its D_{crys} is “as large as possible”, as if M arose from the comparison isomorphism), the object $M^{\psi=1}$ is related to the Galois cohomology H^1 of each of the twists of M in its cyclotomic deformation (see Corollary 7.9 below), and $M^{\psi=0}$ is related to the D_{crys} of the same twists of M . As shown by Berger [4] (and generalized by Nakamura [43]), explicit formulas for Bloch–Kato’s and Perrin-Riou’s exponential maps, and the “ $\delta(M)$ ” formula for the determinant of the latter, follow from a study of the relationship between these two objects. We will therefore focus on describing corresponding objects made from \tilde{M} in $\widetilde{\Phi\Gamma}_{K, A}$ for general K, A, M .

Definition 7.5. For $M \in \Phi\Gamma_{K, A}$, let $C_\psi(M)$ denote the complex

$$0 \rightarrow M \xrightarrow{\psi-1} M \rightarrow 0$$

with the nonzero terms placed in degrees 1 and 2. Denote by $H_\psi^i(M)$ the cohomology groups of this complex.

Let $C_{\psi, \Gamma}(M)$ denote the total complex associated to the double complex

$$0 \rightarrow C(\Gamma, M) \xrightarrow{\psi-1} C(\Gamma, M) \rightarrow 0.$$

Denote by $H_{\psi, \Gamma}^i(M)$ the cohomology groups of this complex. The diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & C(\Gamma, M) & \xrightarrow{\varphi-1} & C(\Gamma, M) & \longrightarrow & 0 \\ & & \downarrow \text{id} & & \downarrow -\psi & & \\ 0 & \longrightarrow & C(\Gamma, M) & \xrightarrow{\psi-1} & C(\Gamma, M) & \longrightarrow & 0 \end{array}$$

induces a morphism $C_{\varphi, \Gamma}(M) \rightarrow C_{\psi, \Gamma}(M)$ which is a quasi-isomorphism [39, Proposition 2.3.6].

Definition 7.6. For $M \in \Phi\Gamma_{K, X}$, $\tilde{M} \in \widetilde{\Phi\Gamma}_{K, X}$ corresponding via Theorem 4.4, define the following sheaves on X :

$$\begin{aligned} \mathcal{H}_{\psi, \Gamma}^i(M) &: \text{Maxspec}(B) \mapsto H_{\psi, \Gamma}^i(M \otimes \mathcal{R}_{K, B}) \\ \mathcal{H}_{\varphi, \Gamma}^i(M) &: \text{Maxspec}(B) \mapsto H_{\varphi, \Gamma}^i(M \otimes \mathcal{R}_{K, B}) \\ \mathcal{H}_{\varphi, \Gamma}^i(\tilde{M}) &: \text{Maxspec}(B) \mapsto H_{\varphi, \Gamma}^i(\tilde{M} \otimes \tilde{\mathcal{R}}_{K, B}). \end{aligned}$$

By Theorem 5.6 and Definition 7.5, the sheaves $\mathcal{H}_{\psi, \Gamma}^i(M)$, $\mathcal{H}_{\varphi, \Gamma}^i(M)$, $\mathcal{H}_{\varphi, \Gamma}^i(\tilde{M})$ are canonically isomorphic; by [39, Theorem 4.4.3, Remark 4.4.4], they are coherent.

Theorem 7.7. For $M \in \Phi\Gamma_{K, A}$, with notation as in Remark 6.3, there is a canonical morphism of complexes

$$C_{\psi, \Gamma}(M') \rightarrow C_\psi(M)$$

which is a quasi-isomorphism.

Proof. Apply [39, Theorem 4.4.8]. □

Corollary 7.8. *Suppose that $M \in \Phi\Gamma_{K,A}$, $\tilde{M} \in \widetilde{\Phi\Gamma}_{K,A}$ correspond as in Theorem 4.4. Then for $X = \text{Maxspec}(A)$, we have canonical isomorphisms*

$$\Gamma(X \times_K W_K, \mathcal{H}_{\psi, \Gamma}^i(\tilde{M} \boxtimes \mathbf{Dfm}_K)) \cong H_{\psi}^i(M).$$

This statement applies to Iwasawa cohomology as follows.

Corollary 7.9. *For $V \in \mathbf{Rep}_A(G_K)$ corresponding to $M \in \Phi\Gamma_{K,A}$, $\tilde{M} \in \widetilde{\Phi\Gamma}_{K,A}$ via Theorem 1.7 and Theorem 4.4, write*

$$H_{\text{Iw}}^i(G_K, V) = \left(\lim_{n \rightarrow \infty} H^i(G_{K(\mu_{p^n})}, T) \right) \otimes_{\mathbb{Z}} \mathbb{Q}$$

for $T \subseteq V$ the unit ball for some Galois-invariant Banach module norm on V (the construction does not depend on the choice), with the transition maps being the corestriction maps. Then for each i , we have functorial isomorphisms

$$H_{\text{Iw}}^i(G_K, V) \hat{\otimes}_{\mathbb{Z}_p[[\Gamma_K]]} \Lambda_K \cong H_{\psi}^i(M) \cong H_{\varphi, \Gamma}^i(\tilde{M} \boxtimes \mathbf{Dfm}_K)$$

of Λ_K -modules compatible with base change.

Proof. Combine Theorem 7.7 with [39, Corollary 4.4.11]. □

Remark 7.10. Corollary 7.9 is a variant of a statement made by Fontaine in his original language of (φ, Γ) -modules; see [14, §II.1] or [4, Theorem II.8].

We now treat the kernel of ψ . Although Theorem 7.12 is ultimately an easy consequence of previous results, its statement is in fact new.

Definition 7.11. Since W_K is a quasi-Stein space, we may write it as the union of an ascending sequence $\{W_{K,n}\}$ of admissible affinoid subspaces. Given an affinoid space X and a coherent sheaf \mathcal{F} on $X \times_K W_K$, define the *module of boundary sections* of \mathcal{F} as

$$\Gamma^{\text{bd}}(\mathcal{F}) = \varinjlim_{n \rightarrow \infty} \Gamma(X \times_K (W_K \setminus W_{K,n}), \mathcal{F}).$$

Theorem 7.12. *Suppose that $M \in \Phi\Gamma_{K,A}$, $\tilde{M} \in \widetilde{\Phi\Gamma}_{K,A}$ correspond as in Theorem 4.4. Then for $X = \text{Maxspec}(A)$, we have canonical isomorphisms*

$$\Gamma^{\text{bd}}(\mathcal{H}_{\varphi, \Gamma}^i(\tilde{M} \boxtimes \mathbf{Dfm}_K)) \cong \begin{cases} M^{\psi=0} & i = 1 \\ 0 & i \neq 1. \end{cases}$$

Proof. The vanishing for $i = 0$ is apparent from Corollary 7.8 (because on a quasi-Stein space, a coherent sheaf is determined by its module of global sections); the vanishing for $i = 2$ follows from the same considerations plus Proposition 7.2. For $i = 1$, the morphism from the left side to the right side is induced by the map $\varphi - 1$ in (7.1.1); to check that it is an isomorphism, we may reduce to the case where A is reduced. In this case, we may use Liu's extension of Tate's Euler characteristic formula [41] (see also [39, Theorem 2.3.11]), applied pointwise on $X \times_K W_K$, to see that the left side is a finite projective module over $A \hat{\otimes}_K \Lambda_K$; we may then use [39, Proposition 4.3.8], applied pointwise on X , to see that the right side is also a finite projective module of the same rank and that the map is an isomorphism. □

8. CODA: BEYOND THE CYCLOTOMIC TOWER

To conclude, we put the previous discussion of the cyclotomic deformation into a context which we find suggestive for future work.

Remark 8.1. Let L be any perfectoid field which is the completion of a Galois algebraic extension of K with Galois group G . By Lemma 2.6, we may also characterize $\widetilde{\Phi\Gamma}_{K,A}$ as the category of objects of $\Phi\mathbf{Mod}_{L,A}$ equipped with continuous semilinear G -actions. For example, we may take $L = \mathbb{C}_K$ to be a completed algebraic closure of K ; in this case, using Theorem 2.18 we get a description of $\Phi\Gamma_{K,A}$ as objects of $\mathbf{BPair}_{\mathbb{C}_K,A}$ equipped with continuous semilinear G_K -actions. In the case $A = \mathbb{Q}_p$, this description is due to Berger [5].

Remark 8.2. In the language of [36], we may view objects of $\widetilde{\Phi\Gamma}_{K,A}$ as sheaves on the pro-étale site of K which are locally finite free modules over the ring sheaf $\mathbb{C}_X \widehat{\otimes}_{\mathbb{Q}_p} A$, equipped with an action of φ .

Remark 8.3. For $M \in \Phi\Gamma_{K,A}$, we have stated descriptions of the objects $M^{\psi=1}$ (in Corollary 7.8) and $M^{\psi=0}$ (in Theorem 7.12) of cyclotomic Iwasawa theory in terms of the corresponding $\tilde{M} \in \widetilde{\Phi\Gamma}_{K,A}$. In light of the previous discussion, the object \tilde{M} can be constructed, and computations can be made with it, without any direct reference to the cyclotomic extension of \mathbb{Q}_p ; the only appearance of the cyclotomic extension in the formulas is via the cyclotomic deformation \mathbf{Dfm}_K on the weight space W_K .

Consequently, for a general p -adic Lie extension L of K with group Γ , one may hope to get something meaningful by forming a suitable deformation space of representations W_K , using the homomorphism $G_K \rightarrow \mathrm{Gal}(L/K) \cong \Gamma$ to define an object of $\mathbf{Rep}_{W_K}(G_K)$, passing to the associated object in $\widetilde{\Phi\Gamma}_{K,W_K}$, taking the external tensor product with \tilde{M} , and considering the cohomology of the result. For starters, in the case where Γ is again a one-dimensional p -adic Lie group, it would be worth comparing this process to other constructions proposed as analogues of [4], e.g., those of Berger–Fourquaux [7] and Schneider–Venjakob [45].

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